Complete toric varieties with reductive automorphism group

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Abstract

We give equivalent and sufficient criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety, to be reductive. In particular we show that the automorphism group of a Gorenstein toric Fano variety is reductive, if the barycenter of the associated reflexive polytope is zero. Furthermore a sharp bound on the dimension of the reductive automorphism group of a complete toric variety is proven by studying the set of Demazure roots.

Introduction

There is an important obstruction to the existence of an Einstein-Kähler metric on a nonsingular Fano variety:

Theorem (Matsushima 1957). *If a nonsingular Fano variety X admits an Einstein-Kähler metric, then* $\mathrm{Aut}(X)$ *is a reductive algebraic group.*

In 1983 Futaki introduced the so called *Futaki character*, whose vanishing is another important necessary condition for the existence of an Einstein-Kähler metric. For a nonsingular toric Fano variety with reductive automorphism group there is an explicit criterion (see [Mab87, Cor. 5.5]):

Theorem (Mabuchi 1987). *Let X be a nonsingular toric Fano variety with* $\mathrm{Aut}(X)$ *reductive.*

The Futaki character of X vanishes if and only if the barycenter of P is zero, where P is the associated reflexive polytope, i.e., the fan of normals of P is associated to X.

In [BS99, Thm. 1.1] Batyrev and Selivanova were able to give a sufficient criterion for the existence of an Einstein-Kähler metric.

Theorem (Batyrev/Selivanova 1999). *Let X be a nonsingular toric Fano variety. We denote by P the associated reflexive polytope.*

If X is symmetric, i.e., the group of lattice automorphisms leaving P invariant has no non-zero fixpoints, then X admits an Einstein-Kähler metric.

In particular they got as a corollary [BS99, Cor. 1.2] that the automorphism group of such a symmetric X is reductive. Expressed in combinatorial terms this just means that the set of lattice points in the relative interiors of facets of P is centrally symmetric. So they asked whether a direct proof for this result exists. Indeed there is even the following generalisation with a simple combinatorial proof (see Theorem 4.2(1)):

Theorem. *Let X be a complete toric variety.*

If the group of automorphisms of the associated fan has no non-zero fixpoints, then $\mathrm{Aut}(X)$ *is reductive.*

Motivated by above results it was conjectured by Batyrev that in the case of a nonsingular toric Fano variety already the vanishing of the barycenter of the associated reflexive polytope were sufficient for the automorphism group to be reductive. Indeed there is even the following more general result that has a purely convex-geometrical proof (see Theorem 4.2(2i)):

Theorem. *Let X be a Gorenstein toric Fano variety.*

If the barycenter of the associated reflexive polytope is zero, then $\mathrm{Aut}(X)$ *is reductive.*

Only very recently Xu-Jia Wang and Xiaohua Zhu could prove that the vanishing of the Futaki character is even sufficient for the existence of an Einstein-Kähler metric in the toric case (see [WZ03, Cor. 1.3]):

Theorem (Wang/Zhu 2003). *Let X be a nonsingular toric Fano variety.*

Then X admits an Einstein-Kähler metric if and only the Futaki character of X vanishes.

Combining the previous results this yields a generalisation of the above theorem of Mabuchi that is also implicit in [WZ03, Lemma 2.2]:

Corollary. *Let X be a nonsingular toric Fano variety.*

Then X admits an Einstein-Kähler metric if and only if the barycenter of P is zero, where P is the associated reflexive polytope.

It is now conjectured by Batyrev that this result may also hold in the singular case of a Gorenstein toric Fano variety.

The paper is organised as follows:

In the first section the notation is fixed and basic definitions are given.

The second section deals with the automorphism group of a d-dimensional complete toric variety X. Here the set of roots $\mathcal{R}$ plays a crucial part in determining the dimension of the identity component and whether the group is reductive (see Prop. 2.2). Using results of Cox in [Cox95] we construct pairwise orthogonal families of roots, so called $\mathcal{S}$-root bases, that parametrize the set of semisimple roots $\mathcal{S}$ in a convenient way, where $\mathcal{S} := \mathcal{R} \cap -\mathcal{R}$. In Prop. 2.18 it is shown that X is isomorphic to a product of projective spaces if and only if there are d linearly independent semisimple roots. When $\mathrm{Aut}(X)$ is reductive, $\dim \mathrm{Aut}^\circ(X) > d^2 - 2$ is a sufficient condition for that (see Theorem 2.21).

In the third section we more closely examine the case of a d-dimensional Gorenstein toric Fano variety X associated to a reflexive polytope P (see [Nil04]). Here a root of X is just a lattice point in the relative interior of a facet of P, so the results of the previous section have a direct geometric interpretation. In

particular we obtain that P has at most $2d$ facets containing roots of P, with equality if and only if X is the product of d projective lines (see Corollary 3.3). Furthermore the intersection of P with the space spanned by all semisimple roots is a reflexive polytope associated to the product of projective spaces (see Theorem 3.9).

In the fourth section we present equivalent and sufficient criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety, to be reductive (see Theorem 4.2).

In the last section we are concerned with d-dimensional centrally symmetric reflexive polytopes. In particular we finish in Theorem 5.3(3) the proof of [Nil04, Thm. 6.4] saying that such a polytope has at most 3^d lattice points, with equality if and only if the associated toric variety is the product of d projective lines.

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1 Notation and basic definitions

In this section we shortly repeat the standard notation for polytopes and toric varieties, as it can be found in [Ewa96], [Ful93] or [Oda88]. In [Bat94] reflexive polytopes were introduced.

Let $N \cong \mathbb{Z}^d$ be a d-dimensional lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^d$ the dual lattice with $\langle \cdot, \cdot \rangle$ the nondegenerate symmetric pairing. As usual, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ and $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^d$ (respectively $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$) will denote the rational (respectively real) scalar extensions.

For a subset S of a real vector space let $\mathrm{lin}(S)$ (respectively $\mathrm{aff}(S)$, $\mathrm{conv}(S)$, $\mathrm{pos}(S)$) be the linear (respectively affine, convex, positive) hull of S. A subset $P \subseteq M_{\mathbb{R}}$ is called a polytope, if it is the convex hull of finitely many points in $M_{\mathbb{R}}$. The boundary of P is denoted by ∂P, the relative interior of P by $\mathrm{relint} P$. When P is full-dimensional, its relative interior is also denoted by $\mathrm{int} P$. A face F of P is denoted by $F \leq P$, the vertices of P form the set $\mathcal{V}(P)$, the facets of P the set $\mathcal{F}(P)$. P is called a lattice polytope, respectively rational polytope, if $\mathcal{V}(P) \subseteq M$, respectively $\mathcal{V}(P) \subseteq M_{\mathbb{Q}}$. An isomorphism of lattice polytopes is an isomorphism of the associated lattices such that the induced real linear isomorphism maps the polytopes onto each other.

We usually denote by $\triangle$ a complete fan in $N_{\mathbb{R}}$. The k-dimensional cones of $\triangle$ form a set $\triangle(k)$. The elements in $\triangle(1)$ are called rays, and given $\tau \in \triangle(1)$, we let v_τ denote the unique generator of $N \cap \tau$.

Let $P \subseteq M_{\mathbb{R}}$ be a rational d-dimensional polytope with $0 \in \mathrm{int} P$. Then we have the important notion of the dual polytope

$$P^* := \{y \in N_{\mathbb{R}} \,:\, \langle x, y \rangle \geq -1 \;\forall\, x \in P\},$$

that is also a rational d-dimensional polytope with $0 \in \mathrm{int} P^*$. The fan $\mathcal{N}_P := \{\mathrm{pos}(F) \,:\, F \leq P^*\}$ is called the normal fan of P.

Duality means $(P^*)^* = P$. There is a natural combinatorial correspondence between i-dimensional faces of P and $(d-1-i)$-dimensional faces of P^* that reverses inclusion.

For a facet $F \leq P$ we let $\eta_F \in N_{\mathbb{Q}}$ denote the uniquely determined inner normal satisfying $\langle \eta_F, F \rangle = -1$. We have

$$\mathcal{V}(P^*) = \{\eta_F \,:\, F \in \mathcal{F}(P)\}.$$

The dual of the product of d_i-dimensional polytopes $P_i \subseteq \mathbb{R}^{d_i}$ with $0 \in \mathrm{int} P_i$ for $i = 1, 2$ is given by

$$(P_1 \times P_2)^* = \mathrm{conv}(P_1^* \times \{0\}, \{0\} \times P_2^*) \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}. \tag{1}$$

By a well-known construction a fan $\triangle$ in $N_{\mathbb{R}}$ defines a toric variety $X := X(N, \triangle)$, i.e., a normal irreducible algebraic variety over $\mathbb{C}$ such that an open embedded algebraic torus $\mathbb{T} = (\mathbb{C}^*)^d$ acts on X in extension of its own action.

Let $P \subseteq M_{\mathbb{R}}$ be a rational polytope. We define the associated toric variety

$$X_P := X(N, \mathcal{N}_P).$$

For d-dimensional rational polytopes P_1, P_2 equation (1) implies

$$X_{P_1} \times X_{P_2} \cong X_{P_1 \times P_2}.$$

Definition 1.1. A complex variety X is called *Gorenstein Fano variety*, if X is projective, normal and its anticanonical divisor is an ample Cartier divisor.

In the toric case there is the following definition (see [Bat94]):

Definition 1.2. A d-dimensional polytope $P \subseteq M_{\mathbb{R}}$ with $0 \in \mathrm{int} P$ is called *reflexive polytope*, if P is a lattice polytope and P^* is a lattice polytope.

Especially reflexive polytopes always appear in dual pairs. There is the following fundamental result (see [Bat94] or [Nil04]):

Theorem 1.3. *Under the map $P \mapsto X_P$ reflexive polytopes correspond uniquely up to isomorphism to Gorenstein toric Fano varieties. There are only finitely many isomorphism types of d-dimensional reflexive polytopes.*

In this context the following definitions are convenient (for a motivation of these notions see [Nil04, Prop. 1.4]).

Definition 1.4. Let $P \subseteq M_{\mathbb{R}}$ be a d-dimensional lattice polytope with $0 \in \mathrm{int} P$.

- P is called a *canonical Fano* polytope, if $\mathrm{int} P \cap M = \{0\}$.
- P is called a *terminal Fano* polytope, if $P \cap M = \{0\} \cup \mathcal{V}(P)$.
- P is called a *smooth Fano* polytope, if the vertices of any facet of P form a $\mathbb{Z}$-basis of the lattice M.

If P is a reflexive polytope, then X_P is a nonsingular toric Fano variety if and only if P^* is a smooth Fano polytope.

The following property [Nil04, Lemma 1.13] characterises reflexive polytopes.

Lemma 1.5. *Let $P \subseteq M_{\mathbb{R}}$ be a reflexive polytope.*

For any $F \in \mathcal{F}(P)$ and $m \in F \cap M$ there is a $\mathbb{Z}$-basis $e_1, \ldots, e_{d-1}, e_d$ of M such that $e_d = m$ and $F \subseteq \{x \in M_{\mathbb{R}} \,:\, x_d = 1\}$; in particular $\eta_F = -e_d^*$ in the dual basis $e_1^*, \ldots, e_d^*$ of N.

Furthermore P is a canonical Fano polytope.

2 The set of roots of a complete toric variety

In this section the set of roots of a complete toric variety is investigated, and some classification results and bounds on the dimension of the automorphism group are achieved.

Throughout the section let $\triangle$ be a complete fan in $N_{\mathbb{R}}$ with associated complete toric variety $X = X(N, \triangle)$.

Definition 2.1. Let $\mathcal{R}$ be the (finite) set of *Demazure roots* of $\triangle$, i.e.,

$$\mathcal{R} := \{m \in M \mid \exists\, \tau \in \triangle(1) : \langle v_\tau, m\rangle = -1 \text{ and } \forall\, \tau' \in \triangle(1)\backslash\{\tau\} : \langle v_{\tau'}, m\rangle \geq 0\}.$$

For $m \in \mathcal{R}$ we denote by η_m the unique primitive generator v_τ of a ray of $\triangle$ with $\langle v_\tau, m\rangle = -1$. For a subset $A \subseteq \mathcal{R}$ it is convenient to define $\eta(A) := \{\eta_m : m \in A\}$.

Let $\mathcal{S} := \mathcal{R} \cap (-\mathcal{R}) = \{m \in \mathcal{R} : -m \in \mathcal{R}\}$ be the set of *semisimple* roots and $\mathcal{U} := \mathcal{R}\backslash\mathcal{S} = \{m \in \mathcal{R} : -m \notin \mathcal{R}\}$ the set of *unipotent* roots. We say that $\triangle$ is *semisimple*, if $\mathcal{R} = \mathcal{S}$, or equivalently $\mathcal{U} = \emptyset$.

Furthermore we define $\mathcal{S}_1 := \{x \in \mathcal{S} : \eta_x \notin \eta(\mathcal{U})\}$ and $\mathcal{S}_2 := \mathcal{S}\backslash\mathcal{S}_1$, analogously $\mathcal{U}_1 := \{x \in \mathcal{U} : \eta_x \notin \eta(\mathcal{S})\}$ and $\mathcal{U}_2 := \mathcal{U}\backslash\mathcal{U}_1$. In particular $\eta(\mathcal{S}_1) \cap \eta(\mathcal{S}_2) = \emptyset$ and $\eta(\mathcal{S}_2) = \eta(\mathcal{U}_2)$.

Usually the set $-\mathcal{R}$ is denoted as the set of Demazure roots (see [Oda88, Prop. 3.13]), however the sign convention here will turn out to be more convenient when considering normal fans of polytopes. Note that $\mathcal{R}$ only depends on the set of rays $\triangle(1)$.

For a root $m \in \mathcal{R}$ we can define a one-parameter subgroup $x_m : \mathbb{C} \to \mathrm{Aut}(X)$. Then the identity component $\mathrm{Aut}^\circ(X)$ is a semidirect product of a reductive algebraic subgroup containing the big torus $(\mathbb{C}^*)^d$ and having $\mathcal{S}$ as a root system and the unipotent radical that is generated by $\{x_m(\mathbb{C}) : m \in \mathcal{U}\}$. Futhermore $\mathrm{Aut}(X)$ is generated by $\mathrm{Aut}^\circ(X)$ and the automorphisms that are induced by lattice automorphisms of the fan $\triangle$. These results are due to Demazure (see [Oda88, p. 140]) in the nonsingular complete case, and were generalised by Cox [Cox95, Cor. 4.7] and Bühler [Büh96]. Bruns and Gubeladze considered the case of a projective toric variety in [BG99, Thm. 5.4]. In particular there is the following result (recall that an algebraic group is reductive, if the unipotent radical is trivial).

Proposition 2.2.

1. $\mathrm{Aut}(X)$ *is reductive if and only if* $\triangle$ *is semisimple.*
2. $\dim \mathrm{Aut}^\circ(X) = |\mathcal{R}| + d$.

When X_P is nonsingular, it is well-known (see [Oda88, p. 140]) that each irreducible component of the root system $\mathcal{S}$ is of type $\mathbf{A}$. Here we will give an explicit description of $\mathcal{S}$ and $\eta(\mathcal{R})$ by orthogonal families of roots that will turn out to be useful for applications.

Definition 2.3. A pair of roots $v, w \in \mathcal{R}$ is called *orthogonal*, in symbols $v \perp w$, if $\langle \eta_v, w\rangle = 0 = \langle \eta_w, v\rangle$. In particular $\eta_{-v} \neq \eta_w \neq \eta_v \neq \eta_{-w}$.

We remark that the term 'orthogonal' may be misleading, because most standard properties do not hold, e.g., $v \perp w$ does not necessarily imply $(-v) \perp w$.

Lemma 2.4. *Let $B = \{b_1, \ldots, b_l\}$ be a non-empty set of roots such that $\langle \eta_{b_i}, b_j \rangle = 0$ for $1 \leq j < i \leq l$. Then B is a $\mathbb{Z}$-basis of $\mathrm{lin}_{\mathbb{R}}(B) \cap M$.*

Proof. We prove the base property by induction on l. Let $x := \lambda_1 b_1 + \cdots + \lambda_l b_l \in M$ with $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$. Then $\lambda_l = -\langle \eta_{b_l}, x \rangle \in \mathbb{Z}$. So $x - \lambda_l b_l = \lambda_1 b_1 + \cdots + \lambda_{l-1} b_{l-1} \in M$. Now proceed by induction. $\square$

We define two special pairwise orthogonal families of roots:

Definition 2.5. Let $A \subseteq \mathcal{R}$.
A pairwise orthogonal family $B \subseteq A$ is called

- *A-facet basis*, if $\eta(A) = \{\eta_b \,:\, b \in B\} \cup \{\eta_{-b} \,:\, b \in B, -b \in A\}$.
- *A-root basis*, if $A = \mathcal{R} \cap \mathrm{lin}(B)$.

When B is an A-root basis, we have $\mathrm{lin}(A) = \mathrm{lin}(B)$, hence $\dim_{\mathbb{R}} \mathrm{lin}(A) = |B|$ by 2.4. If furthermore $B \subseteq \mathcal{S}$, then Prop. 2.10 below implies $A \subseteq \mathcal{S}$ and that B is also an A-facet basis. Note that an $\mathcal{S}$-root basis is *not* a fundamental system for the root system $\mathcal{S}$ in the usual sense.

For an unambiguous description it is convenient to define an equivalence relation on the set of semisimple roots.

Definition 2.6. Let $v \equiv w$ (v is *equivalent* to w), if $v, w \in \mathcal{S}$, $v \neq w$ and $\eta_{-v} = \eta_{-w}$. In particular this yields $\langle \eta_{-v}, w \rangle = -\langle \eta_{-v}, -w \rangle = 1$.

The goal of this section is to explicitly show how to get $\mathcal{S}$-root bases. To this end an algebraic-geometric approach due to Cox shall be discussed:

In [Cox95] Cox described $\mathcal{R}$ as a set of *ordered* pairs of monomials in the homogeneous coordinate ring of the toric variety. For this we denote by $S := \mathbb{C}[x_\rho \,:\, \rho \in \triangle(1)]$ the homogeneous coordinate ring of X, i.e., S is just a polynomial ring where any monomial in S is naturally graded by the class group $\mathrm{Cl}(X)$, i.e., the degree of a monomial $\prod_\rho x_\rho^{k_\rho}$ is the class of the Weil divisor $\sum_\rho k_\rho \mathcal{V}_\rho$, where $\mathcal{V}_\rho$ is the torus-invariant prime divisor corresponding to the ray ρ. Recall that each $\rho \cap N$ is generated by v_ρ.

We let Y denote the set of indeterminates $\{x_\rho \,:\, \rho \in \triangle(1)\}$ and $\mathcal{M}$ the set of monomials in S. For any root $m \in \mathcal{R}$ we define $\rho_m := \mathrm{pos}(\eta_m) \in \triangle(1)$ and $x_m := x_{\rho_m} \in Y$. Now there is the following fundamental result [Cox95, Lemma 4.4] (with a different sign convention):

Lemma 2.7 (Cox 95). *In this notation there is a well-defined bijection*

$$h \,:\, \mathcal{R} \to \{(x_\rho, f) \in Y \times \mathcal{M}, \,:\, x_\rho \neq f, \deg(x_\rho) = \deg(f)\},$$

$$m \mapsto (x_m, \prod_{\rho' \neq \rho_m} x_{\rho'}^{\langle v_{\rho'}, m \rangle}).$$

For $m \in \mathcal{R}$ we have

$$m \in \mathcal{S} \Longleftrightarrow h(m) \in Y \times Y,$$

in this case $h(m) = (x_m, x_{-m})$.

The next result can be used to 'orthogonalise' pairs of roots:

Lemma 2.8. *Let $v, w \in \mathcal{R}$, $v \neq -w$, $\langle \eta_v, w \rangle > 0$. Then $\langle \eta_w, v \rangle = 0$, $v + w \in \mathcal{R}$ and $\eta_{v+w} = \eta_w$. Furthermore*

$$v + w \in \mathcal{S} \text{ iff } v \in \mathcal{S} \text{ and } w \in \mathcal{S}.$$

Proof. Let v correspond to (x_v, f) and w to (x_w, g) as in Lemma 2.7. It is $x_v \neq x_w$. The assumption implies that x_v appears in the monomial g. Assume $\langle \eta_w, v \rangle > 0$. Then x_w would appear in the monomial f. However since $v \neq -w$ this is a contradiction to the antisymmetry of the order relation defined in [Cox95, Lemma 1.3]. The remaining statements are easy to see.

□

Corollary 2.9. *$v \in \mathcal{U}$ and $w \in \mathcal{S}_1$ implies $\langle \eta_v, w \rangle = 0$.*

Lemma 2.8 is a generalisation of parts of [BG02, Prop. 3.3] in a recent paper on polytopal linear groups due to Bruns and Gubeladze. The setting there is that of so called 'column structures' of polytopes where 'column vectors' correspond to roots. Most parts of this lemma were however already independently known and proven by the author as an application of Corollary 3.6 below in the case of a reflexive polytope.

Proposition 2.10. *Let $A \subseteq \mathcal{R}$ and $B \subseteq \mathcal{S}$ an A-root basis partitioned into t equivalence classes of order $c_1, \ldots, c_t$. Then:*

$A = \{\pm b \,:\, b \in B\} \cup \{b - b' \,:\, b, b' \in B,\, b \neq b',\, b \equiv b'\} \subseteq \mathcal{S}$,

$|A| = |B| + \sum_{i \in I} c_i^2 \leq |B| + |B|^2$,

$\eta(A) = \{\eta_{\pm b} \,:\, b \in B\}$, $|\eta(A)| = |B| + t \leq 2|B|$.

Proof. Only the first equation has to be proven: Let $m \in A$, by 2.4 we have $m = \sum_{b \in B} \lambda_b b$ for $\lambda_b \in \mathbb{Z}$. Let $l := \sum_{b \in B} |\lambda_b|$. Proceed by induction on l, let $l > 1$. By orthogonality we have $-1 \leq \langle \eta_b, m \rangle = -\lambda_b$, hence $\lambda_b \leq 1$ for all $b \in B$. Assume there is an element $b \in B$ with $\lambda_b < 0$. Lemma 2.8 implies $b + m \in \operatorname{lin}(B) \cap \mathcal{R} = A$, so $b + m \in \mathcal{S}$ by induction hypothesis. Now Lemma 2.8 yields $-m \in A$. This yields $\lambda_b = -1$. Therefore $\lambda_b \in \{1, 0, -1\}$ for all $b \in B$. Assume $l > 2$. By possibly replacing m with $-m$ we can achieve that there are two elements $b, b' \in B$ with $\lambda_b = 1 = \lambda_{b'}$, hence $\eta_b = \eta_m = \eta_{b'}$, a contradiction. Therefore $l = 2$, and there are two elements $b, b' \in B$ with $m = b - b'$. Assume $b \not\equiv b'$. Then necessarily $\langle \eta_{-b'}, b \rangle = 0$, so $\eta_b = \eta_m = \eta_{-b'}$, a contradiction.

□

Definition 2.11. The grading of the polynomial ring $S := \mathbb{C}[x_\rho \,:\, \rho \in \triangle(1)]$ by the class group $\mathrm{Cl}(X)$ induces a partition of Y into equivalence classes.

1. Let $Y_1, \ldots, Y_p$ be the equivalence classes of order at least two such that there exists no monomial in $\mathcal{M} \backslash Y$ of the same degree.
2. Let $Y_{p+1}, \ldots, Y_q$ be the remaining classes of order at least two.
3. Let $Y_{q+1}, \ldots, Y_r$ be the the equivalence classes of order one such that there exists an monomial in $\mathcal{M} \backslash Y$ of the same degree.
4. Let $Y_{r+1}, \ldots, Y_s$ be the remaining classes of order one.

By Lemma 2.7 ordered pairs of indeterminates contained in one of the equivalence classes $Y_1, \dots, Y_p$ correspond to roots in $\mathcal{S}_1$, ordered pairs in $Y_{p+1}, \dots, Y_q$ correspond to roots in $\mathcal{S}_2$. As changing $m \leftrightarrow -m$ for $m \in \mathcal{S}$ just means to reverse the corresponding pair of monomials, we immediately see that *no element in $\mathcal{S}_1$ is equivalent to an element in $\mathcal{S}_2$*. We have:

$$p = |\,\eta(\mathcal{S}_1)\,|,\ q - p = |\,\eta(\mathcal{S}_2)\,|) = |\,\eta(\mathcal{U}_2)\,|,\ r - q = |\,\eta(\mathcal{U}_1)\,|,\ r = |\,\eta(\mathcal{R})\,|.$$

We get from Lemma 2.7:

$$|\,\mathcal{S}_1\,| = \sum_{i=1}^{p} |\,Y_i\,|(|\,Y_i\,| - 1), \quad |\,\mathcal{S}_2\,| = \sum_{i=p+1}^{q} |\,Y_i\,|(|\,Y_i\,| - 1).$$

Moreover if we define for $i = p+1, \dots, r$ the equivalence class $\mathcal{M}_i$ consisting of monomials in $\mathcal{M}\backslash Y$ having the same degree as an element in Y_i, we get:

$$|\mathcal{U}_1\,| = \sum_{i=q+1}^{r} |\,\mathcal{M}_i\,|, \quad |\mathcal{U}_2\,| = \sum_{i=p+1}^{q} |\,Y_i\,||\,\mathcal{M}_i\,|.$$

In particular $|\mathcal{U}_2\,| \neq \emptyset$ implies $|\mathcal{U}_2\,| \geq 2$. Since by Lemma 2.7 for $i = p+1, \dots, r$ no indeterminate in Y_i can appear in an monomial in $\mathcal{M}_i$, we obtain that $v, w \in \mathcal{U}$ with $\eta_v \neq \eta_w$ and $\deg(x_v) = \deg(x_w)$ are orthogonal. See Example 2.16 below for an illustration.

It is now simple to construct root bases:

Proposition 2.12. *Let a subset $I \subseteq \{1, \dots, q\}$ be given. Choose for any element $i \in I$ a subset $K_i \subseteq Y_i$ of cardinality $c_i + 1$. Denote by R_i a set of c_i semisimple roots corresponding to ordered pairs in K_i with the same fixed second element. Define $B := \cup_{i \in I} R_i$ and $A := \operatorname{lin}(B) \cap \mathcal{R}$.*

Then B is an A-root basis partitioned into equivalence classes $\{R_i\}_{i \in I}$, and any root in A corresponds exactly to an ordered pair in K_i for some $i \in I$.

Moreover any A-basis is given by this construction.

Proof. By construction and Lemma 2.7 $\langle \eta_v, w \rangle = 0 = \langle \eta_w, v \rangle$ for $v, w \in B$, $v \neq w$, hence B is an A-root basis with given equivalence classes. Using Lemma 2.7 and the description of A in Prop. 2.10 the remaining statements are easy to see.

□

Choosing $I = \{1, \dots, q\}$ and $K_i = Y_i$, so $c_i = |\,Y_i\,| - 1$ for $i \in I$, we get:

Corollary 2.13. *$\mathcal{S}$-root bases exist, in particular $\mathcal{R} \cap \operatorname{lin}(\mathcal{S}) = \mathcal{S}$. Moreover*

$$\dim_{\mathbb{R}} \operatorname{lin}(\mathcal{S}) = \sum_{i=1}^{q} (|\,Y_i\,| - 1).$$

Remark 2.14. It is interesting to note that $\operatorname{conv}(\mathcal{S})$ is a centrally symmetric, terminal, reflexive polytope. More precisely due to 2.10 and [DHZ01, proof of Thm. 3.21] there is an isomorphism of lattice polytopes (with respect to lattices $\operatorname{lin}(\mathcal{S}) \cap M$ and $\mathbb{Z}^{c_1 + \cdots + c_q}$)

$$\operatorname{conv}(\mathcal{S}) \cong (\mathcal{Z}_{c_1} \oplus \cdots \oplus \mathcal{Z}_{c_q})^*,$$

where $c_1, \dots, c_q$ are defined as before, and $\mathcal{Z}_n := \mathrm{conv}([0,1]^n, -[0,1]^n)$ is the n-dimensional standard zonotope. For a stronger statement see Theorem 3.9.

Example 2.15. Let's look at $X = \mathbb{P}^d$: We let E_d denote the d-dimensional simplex $\mathrm{conv}(e_1, \dots, e_d, -e_1 - \dots - e_d)$, where $e_1, \dots, e_d$ is a $\mathbb{Z}$-basis of N. Hence E_d is the smooth Fano polytope corresponding to d-dimensional projective space $\mathbb{P}^d$. For $X = \mathbb{P}^d$ and $e_1^*, \dots, e_d^*$ the dual basis of M the family $b_1 := e_1^*$, $b_2 := e_1^* - e_2^*$, $\dots$, $b_d := e_1^* - e_d^*$ forms an $\mathcal{S}$-root basis, where all elements are mutually equivalent. The homogeneous coordinate ring $S = \mathbb{C}[x_0, \dots, x_n]$ is trivially graded. $\mathbb{P}^d$ is semisimple with $d^2 + d$ roots.

Example 2.16. For another example we consider the three-dimensional reflexive simplex $P := \mathrm{conv}((1,0,0),(1,3,0),(1,0,3),(-5,-6,-3))$ with $\mathcal{V}(P^*) = \{(-1,0,0),(-1,0,2),(2,-1,-1),(-1,1,0)\}$. We have $\dim_{\mathbb{R}} \mathcal{S} = 2$, $|\,\mathcal{S}\,| = 4$. F_1 and F_2 contain one antipodal pair of semisimple roots, while F_3 and F_4 contain the other pair. F_3, F_4 each contain three unipotent roots, pairs of unipotent roots in different facets are orthogonal. We can read this off the data $S = \mathbb{C}[x_0, x_1, x_2, x_3]$, $\mathrm{Cl}(X_P) \cong \mathbb{Z}$, $\deg(x_1) = \deg(x_2) = 1$ and $\deg(x_3) = \deg(x_4) = 2$. Hence $Y_1 = \{x_1, x_2\}$, $Y_2 = \{x_3, x_4\}$, $p = 1$, $q = r = s = 2$, $c_1 = 1 = c_2$, $\mathcal{M}_2 = \{x_1^2, x_1 x_2, x_2^2\}$. X_P is just the weighted projective space with weights $(1,1,2,2)$.

Using above results we can show the existence of two special orthogonal families of roots (proof is left to the reader):

Proposition 2.17.

1. *There exists an $\mathbb{R}$-linearly independent family B of roots that can be partitioned into three pairwise disjoint subsets B_1, B_2, B_3 such that B_1 is an $\mathcal{S}_1$-root basis, B_2 is an $\mathcal{S}_2$-root basis, $B_1 \cup B_2$ is an $\mathcal{S}$-root basis and B_3 is a $\mathcal{U}_1$-facet basis such that $\langle \eta_b, b' \rangle = 0$ for all $b \in B_1 \cup B_2$ and $b' \in B_3$.*

   Hence $\dim_{\mathbb{R}} \mathrm{lin}(\mathcal{S}) + |\,\eta(\mathcal{U}_1)\,| = |\,B\,| \leq d$.

2. *There exists an $\mathcal{R}$-facet basis D that can be partitioned into three pairwise disjoint subsets D_1, D_2, D_3 such that D_1 is a $\mathcal{U}_1$-facet basis, D_2 is a $\mathcal{U}_2$-facet basis, $D_1 \cup D_2$ is a $\mathcal{U}$-facet basis and D_3 is an $\mathcal{S}_1$-root basis.*

   Hence $|\,\eta(\mathcal{U}_1)\,| + |\,\eta(\mathcal{U}_2)\,| + \dim_{\mathbb{R}} \mathrm{lin}(\mathcal{S}_1) = |\,D\,| \leq d$.

There exists a classification result:

Proposition 2.18. *A d-dimensional complete toric variety is isomorphic to a product of projective spaces iff there are d linearly independent semisimple roots.*

In this case

$$X \cong \mathbb{P}^{|Y_1|-1} \times \dots \times \mathbb{P}^{|Y_q|-1}.$$

Proof. Let $q = 1$, so there is an $\mathcal{S}$-root basis $b_1, \dots, b_d$ with $\eta_{-b_1} = \dots = \eta_{-b_d}$. Assume there exists $\rho \in \triangle(1)$ with $\rho \notin \{\rho_{b_1}, \dots, \rho_{b_d}, \rho_{-b_1}\}$. Then $\langle v_\rho, b_i \rangle = 0$ for $i = 1, \dots, d$, since $b_i \in \mathcal{S}$. This implies $v_\rho = 0$, a contradiction. Therefore $\triangle(1)$ is determined. Since no cone in $\triangle$ contains a linear subspace, this already implies $X \cong \mathbb{P}^d$. The general case is left to the reader.

□

Corollary 2.19.

1. *$|\eta(\mathcal{U})| \leq d$, $|\eta(\mathcal{U})\backslash\eta(\mathcal{S})| \leq \mathrm{codim}_{\mathbb{R}}\mathrm{lin}(\mathcal{S})$.*

2. *$|\eta(\mathcal{R})| \leq 2d$, with equality iff $X \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.*

3. *$|\mathcal{S}| \leq d^2 + d$, with equality iff $X \cong \mathbb{P}^d$.*

Proof. 1. Follows from 2.17(2).

2. Let D be the $\mathcal{R}$-facet basis from 2.17(2), we have $|D| \leq d$. By definition $\eta(\mathcal{R}) = \{\eta_x \, : \, x \in D_1 \cup D_2\} \cup \{\eta_{\pm x} \, : \, x \in D_3\}$, this gives the upper bound. Equality implies $D = D_3$, i.e., $\mathcal{R} = \mathcal{S}$, with no element in D equivalent to any other. Applying the previous proposition we get the desired result.

3. Follows immediately from Corollary 2.13, Prop. 2.10 and Prop. 2.18.

□

While the case when $M_{\mathbb{R}}$ is spanned by semisimple roots is completely classified, there are at least some partial results in the case of codimension one.

Proposition 2.20. *Let $\dim_{\mathbb{R}} \mathrm{lin}(\mathcal{S}) = d - 1$.*

1. *If $|\triangle(1)| \neq \eta(\mathcal{S})$, then there exists $\tau \in \triangle(1)$ such that $\{\tau\} \subseteq \triangle(1)\backslash\eta(\mathcal{S}) \subseteq \{\pm\tau\}$, and we have $\mathcal{V}_\tau \cong \mathbb{P}^{|Y_1|-1} \times \cdots \times \mathbb{P}^{|Y_q|-1}$.*

2. *If $q = 1$, i.e., $|\mathcal{S}| = d^2 - d$, then $|\eta(\mathcal{U})| = 1$ and $\eta(\mathcal{S}) \cap \eta(\mathcal{U}) = \emptyset$.*

Proof. Let $b_1, \ldots, b_{d-1}$ be an $\mathcal{S}$-root basis. By 2.4 we can find a lattice point $b_d \in M$ such that $b_1, \ldots, b_d$ is an $\mathbb{Z}$-basis of M. Let $e_1, \ldots, e_d$ denote the dual $\mathbb{Z}$-basis of N.

1. Let $\tau \in \triangle(1)\backslash\eta(\mathcal{S})$. Then $\langle v_\tau, b_i \rangle = 0$ for all $i = 1, \ldots, d-1$, hence $v_\tau \in \{\pm e_d\}$. The set $\mathcal{S}$ is by construction canonically the set of roots of $\mathcal{V}_\tau$, so we can apply Prop. 2.18.

2. Let $q = 1$. By 2.10 this is equivalent to $|\mathcal{S}| = (d-1)^2 + d - 1 = d^2 - d$. For $i = 1, \ldots, d-1$ there exist $k_i \in \mathbb{Z}$ such that $\eta_i := \eta_{b_i} = -e_i + k_i e_d$. There exists $k_d \in \mathbb{Z}$ such that $\eta_d := \eta_{-b_1} = e_1 + \cdots + e_{d-1} + k_d e_d$.

Since $|\eta(\mathcal{S})| = d$, there exists $\tau \in \triangle(1)\backslash\eta(\mathcal{S})$, we may assume $v_\tau = e_d$. Let $x = \lambda_1 b_1 + \cdots + \lambda_d b_d \in M$. We have $x \in \mathcal{R}$ with $\eta_x = e_d$ iff $\langle x, e_d \rangle = \lambda_d = -1$ and $\langle x, \eta_i \rangle \geq 0$ for $i = 1, \ldots, d$. This is equivalent to $\lambda_d = -1$, $\lambda_i \leq -k_i$ for $i = 1, \ldots, d-1$ and $\lambda_1 + \cdots + \lambda_{d-1} \geq k_d$. Hence there exists a root $x \in \mathcal{R}$ with $\eta_x = e_d$ if and only if $k_1 + \cdots + k_d \leq 0$.

On the other hand let $u := k_1 b_1 + \cdots + k_{d-1} b_{d-1} + b_d \in M$. Then $u^\perp$ is a hyperplane spanned by $\eta_1, \ldots, \eta_{d-1}$. We have $\langle u, e_d \rangle = 1$ and $\langle u, \eta_d \rangle = k_1 + \cdots + k_d$. Therefore if $|\triangle(1)| = d + 1$, we get $\langle u, \eta_d \rangle < 0$, so there exists $x \in \mathcal{R}$ with $\eta_x = e_d$, necessarily $e_d \in \eta(\mathcal{U})$. Otherwise if $\triangle(1)\backslash\eta(\mathcal{S}) = \{\pm e_d\}$, the analogous computation for $-e_d$ yields that either e_d or $-e_d$ is in $\eta(\mathcal{U})$.

Assume $\eta(\mathcal{S}) \cap \eta(\mathcal{U}) \neq \emptyset$, so $\mathcal{S}_2 \neq \emptyset$. Use the family B in Prop. 2.17(1): Since by assumption all elements in $B_1 \cup B_2$ are mutually equivalent, however no element in B_1 is equivalent to one in B_2, we have $B = B_2$, i.e., $\mathcal{S} = \mathcal{S}_2$. This yields $|\eta(\mathcal{U}_2)| = d$. Since $|\eta(\mathcal{U}_1)| = 1$, we get a contradiction to 2.19(1).

□

For Gorenstein toric Fano varieties the second point cannot simply be improved as can be seen from Example 2.16. Finally we obtain:

Theorem 2.21. *Let X be a d-dimensional complete toric variety with reductive automorphism group. Let $n := \dim \mathrm{Aut}^\circ(X)$. Then*

$$n \leq d^2 + 2d, \quad \textit{with equality only in the case of projective space.}$$

If X is not a product of projective spaces, then

$$n \leq d^2 - 2.$$

Proof. Using 2.2 we see that the first statement is just 2.19(3). For the second statement use 2.18 to get $\dim_{\mathbb{R}} \mathcal{S} \leq d-1$, by 2.10 we have in particular $|\mathcal{S}| \leq (d-1)^2 + d - 1 = d^2 - d$. However equality cannot happen due to 2.20(2), since $\triangle$ is semisimple. Since $|\mathcal{S}|$ is even, we get $|\mathcal{S}| \leq d^2 - d - 2$. Now use 2.2. $\square$

3 The set of roots of a reflexive polytope

Throughout the section let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.

In this section we will focus on Gorenstein toric Fano varieties, these varieties correspond to reflexive polytopes as described in the first section. When P is reflexive, we have by definition that *the set of roots $\mathcal{R}$ of the normal fan $\mathcal{N}_P$ is exactly the set of lattice points in the relative interior of facets of P*.

Definition 3.1. The set $\mathcal{R}$ of roots of P is defined as the set of roots of $\mathcal{N}_P$. For $m \in \mathcal{R}$ we denote by $\mathcal{F}_m$ the unique facet of P that contains m, and we again define $\eta_m = \eta_{\mathcal{F}_m}$ to be the unique primitive inner normal with $\langle \eta_m, \mathcal{F}_m \rangle = -1$. For a subset $A \subseteq \mathcal{R}$ it is convenient to define $\mathcal{F}(A) := \{\mathcal{F}_m : m \in A\}$. We say P is *semisimple*, if $\mathcal{N}_P$ is semisimple, i.e., $\mathcal{R} = -\mathcal{R}$.

Most results of the previous section have now a direct geometric interpretation. Here three examples shall be explicitly stated.

Corollary 3.2. *If a facet of a reflexive polytope contains an unipotent root and a semisimple root x, then the facet containing $-x$ also contains an unipotent root.*

This follows from the fact that $\pm x$ corresponds to a pair of elements in one of the equivalence classes $Y_{p+1}, \ldots, Y_q$ (see 2.11). Alternatively use 2.9.

Corollary 3.3. *Let P be a d-dimensional reflexive polytope.*

Then there are at most $2d$ facets containing lattice points in their relative interior, equality holds iff $P \cong [-1,1]^d$ (isomorphic as lattice polytopes).

This follows from 2.19(2). For another example we apply Prop. 2.18 and Prop. 2.20(2) to $d = 2$ and get a characterisation of semisimple reflexive polygons without using the existing classification (e.g., [Nil04, Prop. 4.1]). The proof relies on the well-known fact that a two-dimensional terminal Fano polytope is a smooth Fano polytope, e.g., [Nil04, Lemma 1.17(1)].

Corollary 3.4. *Let P be a two-dimensional reflexive polytope. For $k \in \mathbb{N}_{>0}$ let the reflexive polytope E_k be defined as in 2.15, i.e., $X_{E_k^*} \cong \mathbb{P}^k$.*

Then P is semisimple iff P is a smooth Fano polytope or $P \cong E_2^*$ or $P \cong E_1^2$.

P or P^* is semisimple iff P or P^* is a smooth Fano polytope.

To sharpen the results of the previous section we need an elementary but fundamental property of pairs of lattice points on the boundary of a reflexive polytope (for a proof see [Nil04, Prop. 3.1]).

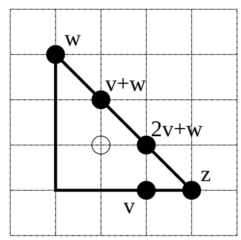
Lemma 3.5. *Let $v, w \in \partial P \cap M$. Then exactly one of the following is true:*

1. *$v \sim w$, i.e., v, w are contained in a common facet*
2. *$v + w = 0$*
3. *$v + w \in \partial P$*

In the last case the following holds:

There exists exactly one pair $(a, b) \in \mathbb{N}^2_{>0}$ with $z := z(v, w) := av + bw \in \partial P$ such that $v \sim z$ and $w \sim z$. We have $a = 1$ or $b = 1$. Any facet containing z (or $v + w$) contains exactly one of the points v or w.

The result shall be illustrated for $P := E_2^*$, i.e., $X_P \cong \mathbb{P}^2$:



This partial addition extends the partial addition of roots in 2.8 (see also [BG02, Def. 3.2]). Extending Definition 2.3 we may also more generally define $v \perp w$ for $v, w \in \partial P \cap M$, if $v + w \in \partial P$ and $z(v, w) = v + w$.

Corollary 3.6. *Let $v \in \mathcal{R}$, $w \in \partial P \cap M$ with $w \notin \mathcal{F}_v$ and $w \neq -v$.*

Then $v + w \in \partial P \cap M$ and $z(v, w) \in \mathcal{F}_v$. Moreover

$$\langle \eta_v, w \rangle > 0 \text{ iff } z(v, w) = av + w \text{ for } a \geq 2.$$

In this case $z(v, w) = (\langle \eta_v, w \rangle + 1)v + w$.

There is a nice property of pairwise orthogonal families of roots:

Proposition 3.7. *Let B be a non-empty set of pairwise orthogonal roots.*

Then $F := \bigcap_{b \in B} \mathcal{F}_b$ is a non-empty face of P of codimension $|B|$, and the sum over all elements in B is a lattice point in the relative interior of F.

Proof. Let $B = \{b_1, \ldots, b_l\}$ with $|B| = l$. For $i \in \{1, \ldots, l\}$ we define $s_i := \sum_{j=1}^{i} b_j$ and $F_i := \cap_{j=1}^{i} \mathcal{F}_{b_j}$. Orthogonality implies that $\{\mathcal{F}_{b_1}, \ldots, \mathcal{F}_{b_l}\}$ is exactly the set of facets containing s_l. Therefore $s_l \in \operatorname{relint} F_l$, and since any l-codimensional face of P is contained in at least l facets, we must have $\operatorname{codim} F_l \leq l$. On the other hand $s_i \notin F_{i+1}$ for all $i = 1, \ldots, l$, so $F_1 \supsetneq \cdots \supsetneq F_l$, hence we obtain $\operatorname{codim} F_l = l$.

□

This yields a corollary concerning facets that contain unipotent roots. The proof follows from the existence of a $\mathcal{U}$-facet basis (see 2.17(2)):

Corollary 3.8. *If $\mathcal{U} \neq \emptyset$, then $\bigcap_{F \in \mathcal{F}(\mathcal{U})} F$ is a face of codimension $|\mathcal{F}(\mathcal{U})| \leq d$.*

In particular if P is not semisimple, then the sum over all lattice points in the non-empty face $\bigcap_{F \in \mathcal{F}(\mathcal{U})} F$ is a non-zero fixpoint of $\mathrm{Aut}_M(P)$.

Now we can improve Prop. 2.18 by taking the ambient space of semisimple roots into account (recall the definition of E_d in 2.15).

Theorem 3.9. *Let $B \subseteq \mathcal{S}$ be an A-root basis for a subset $A \subseteq \mathcal{R}$, and $C_1, \ldots, C_t$ the partition of B into equivalence classes of order $c_1, \ldots, c_t$. Then there are isomorphisms of lattice polytopes (with respect to lattices $\mathrm{lin}(A) \cap M$ and $\mathbb{Z}^{c_1 + \cdots + c_t}$)*

$$P \cap \mathrm{lin}(A) \;\cong\; \bigoplus_{i=1}^{t} P \cap \mathrm{lin}(C_i) \;\cong\; \bigoplus_{i=1}^{t} E^*_{c_i}.$$

In particular the intersection of P with the space spanned by all semisimple roots is again a reflexive polytope corresponding to a product of projective spaces.

Proof. Let $t = 1$, i.e., all elements in B are mutually equivalent. The general case is left to the reader. Let $l = |B| \geq 2$, $B = \{b_1, \ldots, b_l\}$, $b := b_1 + \cdots + b_l$.

Claim: $P \cap \mathrm{lin}(b_1, \ldots, b_l) = \mathrm{conv}(b, b - (l+1)b_i \,:\, i = 1, \ldots, l) \cong E^*_l$.

Denote by Q the simplex on the right hand side of the claim, so $Q \cong E^*_l$.

By 3.7 $b \in \bigcap_{i=1}^{l} \mathcal{F}_{b_i}$. Since by assumption $\langle \eta_{-b_i}, b \rangle = \sum_{j=1}^{l} \langle \eta_{-b_i}, b_j \rangle = \sum_{j=1}^{l} \langle \eta_{-b_j}, b_j \rangle = l$, it follows from 3.6 that $z(-b_i, b) = b - (l+1)b_i \in \mathcal{F}_{-b_i}$ for $i = 1, \ldots, l$. Hence $Q \subseteq P \cap \mathrm{lin}(b_1, \ldots, b_l)$. On the other hand the previous calculation and orthogonality also implies that $Q \cap \mathcal{F}_{b_1}, \ldots, Q \cap \mathcal{F}_{b_l}, Q \cap \mathcal{F}_{-b_1}$ are exactly the facets of the simplex Q. This proves the claim.

□

4 Criteria for a reductive automorphism group

In this section we give several criteria for the automorphism group of a complete toric variety, respectively a Gorenstein toric Fano variety, to be reductive.

Definition 4.1. For a polytope $Q \subseteq M_{\mathbb{R}}$ we let b_Q denote the *barycenter* of Q. When Q is a lattice polytope, we denote by $\mathrm{rvol}(Q)$ the *relative lattice volume* of Q, i.e., $\mathrm{rvol}(\Pi) = 1$ for a fundamental paralleloped Π of the lattice $\mathrm{aff}(Q) \cap M$.

Theorem 4.2.

1. *Let $X = X(N, \triangle)$ be a complete toric variety.*

   The following conditions are equivalent*:*

   (a) $\triangle$ *is semisimple, i.e.,* $\mathrm{Aut}(X)$ *is reductive*

   (b) $\sum_{x \in \mathcal{R}} x = 0$

   (c) $\sum_{\tau \in \triangle(1)} \langle v_\tau, x \rangle = 0$ *for all* $x \in \mathcal{R}$

   If $\sum_{\tau \in \triangle(1)} v_\tau = 0$*, then* $\triangle$ *is semisimple.*

2. *Let X_P be a Gorenstein toric Fano variety for $P \subseteq M_{\mathbb{R}}$ reflexive.*

   The following conditions are equivalent*:*

   (a) *P is semisimple, i.e.,* $\mathrm{Aut}(X_P)$ *is reductive*
   (b) $\sum_{x \in \mathcal{R}} x = 0$
   (c) $\sum_{v \in V(P^*)} \langle v, x \rangle = 0$ *for all* $x \in \mathcal{R}$
   (d) $\sum_{y \in P^* \cap N} \langle y, x \rangle = 0$ *for all* $x \in \mathcal{R}$
   (e) $\langle b_{P^*}, x \rangle = 0$ *for all* $x \in \mathcal{R}$
   (f) $\mathrm{rvol}(F') = \mathrm{rvol}(\mathcal{F}_x)$ *for all* $x \in \mathcal{R}$, $F' \in \mathcal{F}(P)$ *with* $\langle \eta_{F'}, x \rangle > 0$
   (g) $|F' \cap M| = |\mathcal{F}_x \cap M|$ *for all* $x \in \mathcal{R}$, $F' \in \mathcal{F}(P)$ *with* $\langle \eta_{F'}, x \rangle > 0$

   Any one of the following conditions is sufficient *for P to be semisimple:*

   i. $b_P = 0$
   ii. $\sum_{m \in P \cap M} m = 0$
   iii. $b_{P^*} = 0$
   iv. $\sum_{y \in P^* \cap N} y = 0$
   v. $\sum_{v \in V(P^*)} v = 0$
   vi. *All facets of P have the same relative lattice volume*
   vii. *All facets of P have the same number of lattice points*

   Condition vi *implies* v*, e.g., if P is a smooth Fano polytope.*

Remark 4.3. Using the classification of d-dimensional reflexive polytopes for $d \leq 4$ due to Kreuzer and Skarke (see [KS98, KS00, KS02]) we found examples showing that in the second part of the theorem the sufficient conditions i. – v. are pairwise independent, i.e., in general no condition implies any other.

Remark 4.4. As explained in the introduction Batyrev and Selivanova obtained in [BS99] from the existence of an Einstein-Kähler metric, that if P is a reflexive polytope, X_P is nonsingular and P is *symmetric*, i.e., the group $\mathrm{Aut}_M(P)$ of linear lattice automorphisms leaving P invariant has only the origin 0 as a fixpoint, then P has to be semisimple. They asked whether a direct proof for this combinatorial result exists. Indeed if P is a symmetric reflexive polytope, we easily see that the second equivalent and even the first five sufficient conditions in the second part of the theorem are satisfied. For yet another proof see also Corollary 3.8. Furthermore the first part of the theorem immediately yields a generalisation to complete toric varieties, for this see the fourth theorem in the introduction.

For the proof of Theorem 4.2 we need some lemmas. The first is just a simple observation:

Lemma 4.5. *Let $\triangle$ be a complete fan in $N_{\mathbb{R}}$.*

$$m \in \mathcal{R} \Longrightarrow \sum_{\tau \in \triangle(1)} \langle v_\tau, m \rangle \in \mathbb{N},$$

in this case

$$m \in \mathcal{S} \Longleftrightarrow \sum_{\tau \in \triangle(1)} \langle v_\tau, m \rangle = 0.$$

Lemma 4.6. *Let $\triangle$ be a complete fan in $N_{\mathbb{R}}$.*
Let $A \subseteq \mathcal{R}$ be a subset such that

$$\sum_{m \in A} k_m m = 0$$

for some positive integers $\{k_m\}_{m \in A}$. Then $A \subseteq \mathcal{S}$.

Proof. Assume $A \cap \mathcal{U} \neq \emptyset$. Then by 4.5
$0 = \sum_{\tau \in \triangle(1)} \langle v_\tau, \sum_{m \in A} k_m m \rangle = \sum_{m \in A \cap \mathcal{U}} k_m \sum_{\tau \in \triangle(1)} \langle v_\tau, m \rangle \geq 1$, a contradiction.

□

In the case of a reflexive polytope the following result is fundamental:

Lemma 4.7. *Let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.*
Let $m \in \mathcal{R}$. Define the canonical projection map along m

$$\pi_m : M_{\mathbb{R}} \to M_{\mathbb{R}}/\mathbb{R}m.$$

Then π_m induces an isomorphism of lattice polytopes

$$\mathcal{F}_m \to \pi_m(P),$$

with respect to the lattices $\operatorname{aff}(F) \cap M$ and $M/m\mathbb{Z}$.

Proof. [Nil04, Prop. 2.2] immediately implies that $\pi_m : \mathcal{F}_m \to \pi_m(P)$ is a bijection. It is even an isomorphism of lattice polytopes by 1.5.

Another proof can be easily done using only the definition of a root.

□

Using this lemma we get a reformulation of 4.5. Note that $A - B := \{a - b : a \in A,\, b \in B\}$ for arbitrary sets $A, B \subseteq \mathbb{R}^d$.

Lemma 4.8. *Let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.*
For $m \in \mathcal{R}$ with $F := \mathcal{F}_m$ we have:

1. *$P \subseteq F - \mathbb{R}_{\geq 0} x$, $P \cap M \subseteq (F \cap M) - \mathbb{N}x$, $\{n \in P^* \cap N : \langle n, m \rangle < 0\} = \{\eta_m\}$.*
2. *$P = \operatorname{conv}(F, F')$ iff there is only one facet F' with $\langle \eta_{F'}, m \rangle > 0$.*
3. *$m \in \mathcal{S}$ iff the previous condition is satisfied and $\langle \eta_{F'}, m \rangle = 1$.*
   In this case $F' = \mathcal{F}_{-m}$. Furthermore F and F' are naturally isomorphic as lattice polytopes and $\{n \in P^* \cap N : \langle n, m \rangle \neq 0\} = \{\eta_m, \eta_{-m}\}$.

Lemma 4.9. *Let P be a d-dimensional reflexive polytope in $M_{\mathbb{R}}$.*

For $v \in \mathcal{V}(P^*)$ we denote by $v^* \in \mathcal{F}(P)$ the corresponding facet of P. Then

$$\sum_{v \in \mathcal{V}(P^*)} \mathrm{rvol}(v^*)\, v = 0.$$

Proof. Having chosen a fixed lattice basis of M we denote by vol the associated differential-geometric volume in $M_{\mathbb{R}} \cong \mathbb{R}^d$. Let $F \in \mathcal{F}(P)$ arbitrary. Since η_F is primitive, it is a well-known fact that the determinant of the lattice $\mathrm{aff}(F) \cap M$, i.e., the volume of a fundamental paralleloped, is exactly $\|\eta_F\|$, hence we get $\mathrm{vol}(F) = \mathrm{rvol}(F) \cdot \|\eta_F\|$. The easy direction of the so called existence theorem of Minkowski (see [BF71, no. 60]) yields $\sum_{F \in \mathcal{F}(P)} \mathrm{rvol}(F)\, \eta_F = 0$.

□

The approximation approach in the next proof is based upon an idea of Batyrev. Note that a facet F of a d-dimensional polytope $Q \subseteq M_{\mathbb{R}}$ is said to be parallel to $\mathbb{R}x$ for $x \in M_{\mathbb{R}}$, if $\langle \eta_F, x \rangle = 0$.

Lemma 4.10. *Let $Q \subseteq M_{\mathbb{R}}$ be a d-dimensional polytope with a facet F and $x \in \mathrm{aff}(F)$ such that $Q \subseteq F - \mathbb{R}_{\geq 0} x$. For $q \in Q$ with $q = y - lx$, where $y \in F$ and $l \in \mathbb{R}_{\geq 0}$, define $a(q) := y - 2lx$. This definition extends uniquely to an affine map a of $M_{\mathbb{R}}$.*

Then $a(b_Q)$ is either in the interior of Q or in the relative interior of a facet of Q not parallel to $\mathbb{R}x$. The last case happens exactly iff there exists only one facet $F' \neq F$ not parallel to $\mathbb{R}x$.

Proof. First assume there is exactly one facet $F' \neq F$ not parallel to $\mathbb{R}x$. This implies $Q = \mathrm{conv}(F, F')$. Choose an $\mathbb{R}$-basis $e_1, \ldots, e_d$ of $M_{\mathbb{R}}$ such that $e_d = x$ and $\mathbb{R}e_1, \ldots, \mathbb{R}e_{d-1}$ are parallel to F. Now let $y \in F$ and define $h(y) \in \mathbb{N}$ such that $y - h(y)x \in F'$. For $k \in \mathbb{N}_{>0}$ let $F_k(y) := y + \cup_{i=1}^{d-1} [-\frac{1}{2k}, \frac{1}{2k}] e_i$ and $Q_k(y) := F_k(y) - [0, h(y)]x$. Then $b_{Q_k(y)} = y - \frac{h(y)}{2} x$ and $a(b_{Q_k(y)}) = y - h(y)x \in F'$. Let $M' := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{d-1}$ and $z \in \mathrm{relint} F$. For any $k \in \mathbb{N}_{>0}$ we define $G_k := (z + \frac{1}{k} M') \cap F$ and $F_k := \cup_{y \in G_k} F_k(y)$. For $k \to \infty$ the sets F_k converge uniformly to F. Therefore also $Q_k := \cup_{y \in G_k} Q_k(y)$ converges uniformly to Q for $k \to \infty$. This implies that b_{Q_k} converges to b_Q for $k \to \infty$. Now a is easily seen to be the restriction of an affine map of $M_{\mathbb{R}}$, hence as b_{Q_k} is a finite convex combination of $\{b_{Q_k(y)} \,:\, y \in G_k\}$ for any $k \in \mathbb{N}_{>0}$, also $a(b_{Q_k})$ is a finite convex combination of $\{a(b_{Q_k(y)} \,:\, y \in G_k\} \subseteq F'$ for any $k \in \mathbb{N}_{>0}$. This implies $a(b_{Q_k}) \in F'$ for any $k \in \mathbb{N}_{>0}$. Since a is continuous and F' is closed, this yields $a(b_Q) \in F'$. Hence obviously $a(b_Q) \in \mathrm{relint} F'$.

Now let there be more than one facet different from F that is not parallel to $\mathbb{R}x$. Then choose a polyhedral subdivision of Q into finitely many polytopes $\{K_j\}$ such that any K_j satifies the condition of the previous case. Then b_Q is a proper convex combination of $\{b_{K_j}\}$, therefore by affinity also $a(b_Q)$ is a proper convex combination of $\{a(b_{K_j})\} \subseteq \partial Q$. However since not all $a(b_{K_j})$ are contained in one facet, $a(b_Q)$ is in the interior of Q.

□

Proof of Theorem 4.2. The first part of the theorem, when X is a complete toric variety, follows from 4.5 and 4.6. So let $X = X_P$ for $P \subseteq M_{\mathbb{R}}$ a d-dimensional reflexive polytope, and we consider the second part of the theorem.

(a) and (b) are equivalent by 4.6. The equivalences of $(a), (c), (d), (e)$ and the sufficiency of iii, iv, v follow from 4.5 and 4.8.

(f) and (g) are necessary conditions for semisimplicity due to 4.8.

Let (f) be satisfied and $x \in \mathcal{R}$. By 4.8(1) and 4.9 we have

$$\operatorname{rvol}(\mathcal{F}_x) = \sum_{v \in \mathcal{V}(P^*),\, \langle v,x \rangle > 0} \operatorname{rvol}(v^*) \langle v, x \rangle.$$

By assumption there is only one vertex $v \in \mathcal{V}(P^*)$ with $\langle v, x \rangle > 0$, furthermore $\langle v, x \rangle = 1$. Hence 4.8 implies $x \in \mathcal{S}$.

Let (g) be satisfied. Let $x \in \mathcal{R}$, $F := \mathcal{F}_x$ and $F' \in \mathcal{F}(P)$ with $\langle \eta_{F'}, x \rangle > 0$. Due to 4.8(1) and by assumption there is a bijective map $h : F' \to F$ of lattice polytopes, i.e., $h(F' \cap M) \subseteq F \cap M$. Now there exists a lattice point $y \in F'$ with $h(y) = m$. Since $P = \operatorname{conv}(F, F')$ and P is a canonical polytope, we obtain $y = -m \in \operatorname{relint} F'$, hence $m \in \mathcal{S}$.

The sufficiency of vi, vii is now trivial, 4.9 shows that vi implies v.

From now on let $x \in \mathcal{R}$ and a the affine map defined as in 4.10 for $Q := P$ and $F := \mathcal{F}_x$.

Let i be satisfied. By 4.8(1) we can apply Lemma 4.10 to get $-x = x - 2x = a(0) = a(b_P) \in \mathcal{R}$, since P is a canonical Fano polytope.

Finally let ii be satisfied. For any $y \in F \cap M$ define $x_y \in P \cap M$ with $x_y := y - kx$ for $k \in \mathbb{N}$ maximal, and let $T_y := [y, x_y]$. Then 4.8(1) implies that

$$\begin{aligned} -x \;=\; a(0) &= a\left(\tfrac{1}{|P \cap M|} \sum_{m \in P \cap M} m\right) = a\left(\sum_{y \in F \cap M} \tfrac{|T_y \cap M|}{|P \cap M|} \tfrac{1}{|T_y \cap M|} \sum_{m \in T_y \cap M} m\right) \\ &= \sum_{y \in F \cap M} \tfrac{|T_y \cap M|}{|P \cap M|} a\left(\tfrac{1}{|T_y \cap M|} \sum_{m \in T_y \cap M} m\right) = \sum_{y \in F \cap M} \tfrac{|T_y \cap M|}{|P \cap M|} x_y. \end{aligned}$$

Hence $-x$ is a proper convex combination of $\{x_y\}_{y \in F \cap M}$. As P is a canonical Fano polytope, we get $-x \in \mathcal{R}$.

$\square$

5 Centrally symmetric reflexive polytopes

In this section the following result is going to be proved (recall the definition of the lattice polytope $E_1 := [-1, 1]$).

Theorem 5.1. *Let P be a centrally symmetric d-dimensional reflexive polytope with X_P the toric variety associated to $\mathcal{N}_P$. Then*

1. *$P \cong E_1^{\frac{|\mathcal{R}|}{2}} \times G$ for a $\frac{|\mathcal{R}|}{2}$-codimensional face G of P that is a centrally symmetric reflexive polytope (with respect to $\operatorname{aff}(G) \cap M$ and a unique lattice point in $\operatorname{relint} G$) and has no roots itself.*

2. *Any facet contains at most 3^{d-1} lattice points and at most one root of P. P contains at most 3^d lattice points and has at most $2d$ roots. Hence*

$$\dim \operatorname{Aut}^\circ(X_P) \le 3d.$$

3. *The following statements are equivalent:*

   (a) *P contains 3^d lattice points*

   (b) *P has $2d$ roots, i.e.,* $\dim \mathrm{Aut}^\circ(X_P) = 3d$

   (c) *Every facet of P contains a root of P*

   (d) *Every facet of P has 3^{d-1} lattice points*

   (e) *$P \cong E_1^d$, i.e., $X_P \cong \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$*

The first property immediately implies (see 2.2):

Corollary 5.2. *Let P be a centrally symmetric reflexive polytope with X_P the toric variety associated to $\mathcal{N}_P$.*

If P contains no facet that is centrally symmetric with respect to a root of P, or there are at most $d-1$ facets of P that can be decomposed as a product of lattice polytopes $E_1 \times F'$, then P has no roots.

Hence if $d \geq 3$ and P is simplicial, or $d \geq 4$ and any facet of P is simplicial, then

$$\dim \mathrm{Aut}^\circ(X_P) = d.$$

For the proof of Theorem 5.1 we need the following lemma that is an easy corollary of 4.8 and 1.5:

Lemma 5.3. *Let P be a centrally symmetric reflexive polytope.*
Let $F \in \mathcal{F}(P)$. Then

$$P \cong E_1 \times F \text{ iff } F \text{ contains a root } x \text{ of } P.$$

In this case F is a centrally symmetric reflexive polytope (with respect to the lattice $\mathrm{aff}(F) \cap M$ *with origin x).*

Proof of Theorem 5.1. 1. Apply the previous lemma inductively.

2. The bounds on the lattice points were proven in [Nil04, Thm. 6.4]. Since as just seen any facet of P containing a root is reflexive, hence a canonical Fano polytope, it contains only one root of P. Now we apply 2.2(2) and 1. (or 3.3).

3. (b) $\Leftrightarrow$ (e) $\Leftrightarrow$ (c): Since P as a centrally symmetric polytope contains at least $2d$ facets, this follows from 1., alternatively use 2. and 3.3.

For the remaining equivalences we need the canonical map

$$\alpha \,:\, P \cap M \to M/3M \cong (\mathbb{Z}/3\mathbb{Z})^d.$$

In the proof of [Nil04, Thm. 6.4] it was shown that α is injective.

Let $F \in \mathcal{F}(P)$ be arbitrary but fixed. Define $u := \eta_F \in \mathcal{V}(P^*)$ and also the $\mathbb{Z}/3\mathbb{Z}$-extended map $\alpha(u) \,:\, M/3M \to \mathbb{Z}/3\mathbb{Z}$. For $m \in P \cap M$ we have $\langle u, m \rangle \in \{-1, 0, 1\}$, in particular

$$m \in F \Longleftrightarrow \langle \alpha(u), \alpha(m) \rangle = -1 \in \mathbb{Z}/3\mathbb{Z}.$$

(c) $\Rightarrow$ (a): Trivial, since (c) $\Leftrightarrow$ (e).

(a) $\Rightarrow$ (d): If P contains 3^d lattice points, then α is a bijection, and therefore $|\,F \cap M\,| = |\,\{z \in M/3M \,:\, \langle \alpha(\eta_F), z \rangle = -1\}\,| = 3^{d-1}$.

(d) $\Rightarrow$ (c): The assumption implies that for any facet $F' \in \mathcal{F}(P)$ the map

$$\alpha|_{F'} :\, F' \cap M \to \{z \in M/3M \,:\, \langle \alpha(\eta_{F'}), z\rangle = -1\}$$

is a bijection. Define $x := (1/3^{d-1}) \sum_{m \in F \cap M} m \in \mathrm{relint} F$.

It remains to prove $x \in M$.

Choose a facet $G \in \mathcal{F}(P^*)$ and an $\mathbb{R}$-linearly independent family $w_1, \ldots, w_d$ of vertices of G such that $w_1 = u$ and $w_2, \ldots, w_d$ are contained in a $(d-2)$-dimensional face of P^*.

Denote the corresponding facets of P by $F_1, F_2, \ldots, F_d$ with $\eta_{F_j} = w_j$ for $j = 1, \ldots, d$, so $F_1 = F$. Then $Q := \cap_{j=2}^d F_j$ is a one-dimensional face of P. Therefore also the affine span of $\alpha(Q \cap M)$ is a one-dimensional affine subspace of $M/3M$. Since $|\,F \cap Q\,| = 1$ there exists an element $b \in M/3M$ such that $\langle \alpha(u), b\rangle = 0$ and $\langle \alpha(w_j), b\rangle = -1$ for all $j = 2, \ldots, d$. Applying the assumption to F_2 yields a lattice point $v \in P \cap M$ with $\alpha(v) = b$. Hence also $\langle u, v\rangle = 0$ and $\langle w_j, v\rangle = -1$ for $j = 2, \ldots, d$.

By 1.5 we find a $\mathbb{Z}$-basis $e_1^* = u, e_2^*, \ldots, e_d^*$ of N such that for any $j = 2, \ldots, d$ there exist $\lambda_{j,k} \in \mathbb{R}$ with $e_j^* = \lambda_{j,2}(w_2 - u) + \cdots + \lambda_{j,d}(w_d - u)$.

$\cdot$ *Fact 1:* $\langle w_k, \sum_{m \in F \cap M} m\rangle = 0$ for $k = 2, \ldots, d$.

(*Proof:* Since $F \cap F_k \neq \emptyset$, the assumption implies for $i = -1, 0, 1 \in \mathbb{Z}/3\mathbb{Z}$: $|\,\{z \in M/3M \,:\, \langle \alpha(u), z\rangle = -1, \langle \alpha(w_k), z\rangle = i\}\,| = 3^{d-2}$.)

$\cdot$ *Fact 2:* $\sum_{k=2}^d \lambda_{j,k} \in \mathbb{Z}$ for $j = 2, \ldots, d$.

(*Proof:* $\langle e_j^*, v\rangle = (-\sum_{k=2}^d \lambda_{j,k})\langle u, v\rangle + \sum_{k=2}^d \lambda_{j,k}\langle w_k, v\rangle = -\sum_{k=2}^d \lambda_{j,k}$ by the choice of v.)

Using these two facts we can finish the proof:

$$\begin{aligned}
\langle e_1^*, x\rangle &= \langle u, x\rangle = -1 \in \mathbb{Z}, \\
\langle e_j^*, x\rangle &= (1/3^{d-1}) \left((-\sum_{k=2}^d \lambda_{j,k})\langle u, \sum_{m \in F \cap M} m\rangle + \sum_{k=2}^d \lambda_{j,k}\langle w_k, \sum_{m \in F \cap M} m\rangle \right) \\
&= \sum_{k=2}^d \lambda_{j,k} \in \mathbb{Z} \;\text{ for } j = 2, \ldots, d.
\end{aligned}$$

Hence $x \in M$.

□

Remark 5.4. Dropping the assumption of reflexivity and regarding just a complete toric variety $X = X(N, \triangle)$ with centrally symmetric $\triangle(1)$ we still get immediately from 2.1, 2.2 and 2.19(2) that $\dim \mathrm{Aut}^\circ(X) \leq 3d$, with equality if and only if $X \cong (\mathbb{P}^1)^d$.

For X as before, assume X is also Gorenstein, i.e., the anticanonical divisor $-K_X$ is a Cartier divisor. In this case we can still show by slightly modifiying the proof of [Nil04, Thm. 6.4] that $h^0(X, -K_X) \leq 3^d$.

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